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A two-mode iterative procedure for data fitting

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1. Introduction

An iterative procedure in the residual space of the l_2 fitting problem is described. The idea is that following a preliminary fitting of the data it may be possible to nominate certain of the data values to be more closely fitted than they are by the preliminary fitting. On the other hand it may be decided that certain data values are too suspect to be used. By operating in one or two modes the iterative procedure will accomplish either of these effects. Other applications may occur to the reader.

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2. The iterative sequence

Given a data set $h_1, h_2, \ldots, h_M$, observed at the discrete positions $x_1, x_2, \ldots, x_M$ it may be that we wish to represent this data set by a functional fitting of the form

$$\underset{M1}{\underline{h}} = b_1 \underset{M1}{\underline{\phi}_1} + b_2 \underset{M1}{\underline{\phi}_2} + \text{-------} + b_n \underset{M1}{\underline{\phi}_n} + \underset{M1}{\underline{r}} \qquad (1)$$

where $\underline{h}$ is the M-vector $\underline{h} \equiv \underline{h}(h_1, h_2, \ldots, h_M)$. $\underline{\phi}_i$ are the orthogonal M-vectors $\underline{\phi}_i \equiv \underline{\phi}_i(\phi_{1i}, \phi_{2i}, \ldots, \phi_{Mi})$, namely the evaluations of the i-th base function over the domain, and

$\underline{r}$ is the M-vector of residuals.

The least-squares solution of the problem is given by

$$\underset{n1}{\underline{\hat{b}}} = \underset{nM}{\underline{\tilde{\Phi}}} \cdot \underset{M1}{\underline{h}} \qquad (2)$$

where $\underline{\hat{b}}$ is the n x 1 coefficient vector and $\underset{Mn}{\underline{\Phi}}$ is the matrix of column vectors

$$\underline{\Phi} = (\underline{\phi}_1 \,|\, \underline{\phi}_2 \,|\, \underline{\phi}_3 \,|\, \text{- - - -} \,|\, \underline{\phi}_n)$$

Having thus obtained the fitting least squares function

$$\underset{M1}{\underline{z}} = \underset{Mn}{\underline{\Phi}} \cdot \underset{n1}{\underline{\hat{b}}} \qquad (3)$$

it may then be possible by graphing the function and comparing its' values at the data points with the actual h_i values, and also possibly by consideration of some quite extraneous information, to form a judgement that certain data values should be much more closely fitted than the rest. This may be accomplished by the following iterative procedure -

Let the fitting (3) by regarded as a starting point and indicate this by adding a subscript o.

$$\underline{z}_o = \underline{\Phi} \cdot \underline{\hat{b}}_o \qquad (4)$$

As the arrangement of the components of the vectors $\underline{h}$, $\underline{\phi}_i$ and $\underline{z}$, does not affect the fitting, provided that the arrangement is consistent throughout all of them, let the data values to which special importance is attached occupy the first m places in $\underline{h}$. Then a correction increment $\underline{\hat{b}}'$ to $\underline{\hat{b}}_o$ may be obtained by solving the partitioned system

$$\left(\begin{array}{c|c} \underset{mm}{\underline{\Phi}_1} & \underset{m(n-m)}{\underline{\Phi}_2} \\ \hline \underset{(M-m)m}{\underline{\Phi}_3} & \underset{(M-m)(n-m)}{\underline{\Phi}_4} \end{array}\right) \cdot \underset{n1}{\underline{\hat{b}}'} = \left(\begin{array}{c} \underset{m1}{\underline{d}_1} \\ \hline \underset{(M-m)1}{\underline{0}} \end{array}\right) = \left(\begin{array}{c} \underset{m1}{\underline{h}} - \underset{m1}{\underline{z}_o} \\ \hline \underset{(M-m)1}{\underline{0}} \end{array}\right) \qquad (5)$$

whence

$$\hat{\underline{b}}' = \widetilde{\begin{pmatrix} \underline{\Phi}_1 & \underline{\Phi}_2 \\ \underline{\Phi}_3 & \underline{\Phi}_4 \end{pmatrix}} \cdot \begin{pmatrix} \underline{d}_1 \\ \underline{0} \end{pmatrix}$$

i.e.

$$\hat{\underline{b}}' = \begin{pmatrix} \widetilde{\underline{\Phi}}_1 & \widetilde{\underline{\Phi}}_3 \\ \widetilde{\underline{\Phi}}_2 & \widetilde{\underline{\Phi}}_4 \end{pmatrix} \cdot \begin{pmatrix} \underline{d}_1 \\ \underline{0} \end{pmatrix}$$

whence

$$\hat{\underline{b}}' = \begin{pmatrix} \widetilde{\underline{\Phi}}_1 \cdot \underline{d}_1 \\ \widetilde{\underline{\Phi}}_2 \cdot \underline{d}_1 \end{pmatrix} \qquad (6)$$

This correction increment $\hat{\underline{b}}'$ can then be added to $\hat{\underline{b}}_0$ to get

$$\hat{\underline{b}}_1 = \hat{\underline{b}}_0 + \hat{\underline{b}}' = \widetilde{\underline{\Phi}} \cdot \underline{z}_0 + \begin{pmatrix} \widetilde{\underline{\Phi}}_1 \cdot \underline{d}_1 \\ \widetilde{\underline{\Phi}}_2 \cdot \underline{d}_1 \end{pmatrix}$$

i.e.

$$\hat{\underline{b}}_1 = \begin{pmatrix} \widetilde{\underline{\Phi}}_1 & \widetilde{\underline{\Phi}}_3 \\ \widetilde{\underline{\Phi}}_2 & \widetilde{\underline{\Phi}}_4 \end{pmatrix} \cdot \begin{pmatrix} \underline{z}_0 \\ \underline{z}_0 \end{pmatrix} + \begin{pmatrix} \widetilde{\underline{\Phi}}_1 \cdot \underline{d}_1 \\ \widetilde{\underline{\Phi}}_2 \cdot \underline{d}_1 \end{pmatrix}$$

The coefficient $\hat{\underline{b}}_1$ is thus split into two parts

$$\begin{pmatrix} \hat{\underline{b}}_1 \\ \hline \hat{\underline{b}}_1 \end{pmatrix} = \begin{pmatrix} \widetilde{\underline{\Phi}}_1 \cdot \underline{z}_0 + \widetilde{\underline{\Phi}}_3 \cdot \underline{z}_0 \\ \widetilde{\underline{\Phi}}_2 \cdot \underline{z}_0 + \widetilde{\underline{\Phi}}_4 \cdot \underline{z}_0 \end{pmatrix} + \begin{pmatrix} \widetilde{\underline{\Phi}}_1 \cdot \underline{d}_1 \\ \widetilde{\underline{\Phi}}_2 \cdot \underline{d}_1 \end{pmatrix}$$

i.e.

$$\begin{pmatrix} \hat{\underline{b}}_1 \\ \hline \hat{\underline{b}}_1 \end{pmatrix} = \begin{pmatrix} \widetilde{\underline{\Phi}}_3 \cdot \underline{z}_0 + \widetilde{\underline{\Phi}}_1 \cdot \underline{h} \\ \widetilde{\underline{\Phi}}_4 \cdot \underline{z}_0 + \widetilde{\underline{\Phi}}_2 \cdot \underline{h} \end{pmatrix} \qquad (7)$$

$\hat{\underline{b}}_1$ can now be used to obtain a new fitting over the whole domain of M points

$$\underset{M1}{\underline{z}_1} = \underset{Mn}{\underline{\Phi}} \cdot \underset{n1}{\hat{\underline{b}}_1}$$

Then the first m values in $\underset{m1}{h}$ can be used again to obtain a second correction increment $\underset{n1}{\hat{b}''}$ by solving

$$\left(\begin{array}{c|c} \Phi_1 & \Phi_2 \\ \hline \Phi_3 & \Phi_4 \end{array}\right) \cdot \underset{n1}{\hat{b}''} = \left(\begin{array}{c} \underset{m1}{d_2} \\ \hline \underset{(M-m)1}{0} \end{array}\right) = \left(\begin{array}{c} \underset{m1}{h} - \underset{m1}{z_1} \\ \hline \underset{(M-m)1}{0} \end{array}\right)$$

to get

$$\underset{n1}{\hat{b}''} = \left(\begin{array}{c} \underset{mm}{\tilde{\Phi}_1} \cdot \underset{m1}{d_2} \\ \hline \underset{(n-m)m}{\tilde{\Phi}_2} \cdot \underset{m1}{d_2} \end{array}\right)$$

This correction increment is then added to $\underset{n1}{\hat{b}_1}$ to get

$$\underset{n1}{\hat{b}_2} = \underset{n1}{\hat{b}_1} + \underset{n1}{\hat{b}''} = \underset{nM}{\tilde{\Phi}} \cdot \underset{M1}{z_1} + \left(\begin{array}{c} \underset{mm}{\tilde{\Phi}_1} \cdot \underset{m1}{d_2} \\ \hline \underset{(n-m)m}{\tilde{\Phi}_2} \cdot \underset{m1}{d_2} \end{array}\right)$$

and by the same manipulations as led to this yields

$$\left(\begin{array}{c} \underset{m1}{\hat{b}_2} \\ \hline \underset{(n-m)1}{\hat{b}_2} \end{array}\right) = \left(\begin{array}{c} \underset{m(M-m)}{\tilde{\Phi}_3} \cdot \underset{(M-m)1}{z_1} + \underset{mm}{\tilde{\Phi}_1} \cdot \underset{m1}{h} \\ \hline \underset{(n-m)(M-m)}{\tilde{\Phi}_4} \cdot \underset{(M-m)1}{z_1} + \underset{(n-m)m}{\tilde{\Phi}_2} \cdot \underset{m1}{h} \end{array}\right) \tag{8}$$

Then $\underset{n1}{\hat{b}_2}$ can be used to find a $\underset{M1}{z_2}$ to get an increment $\underset{n1}{\hat{b}'''}$ and so on. At the i-th iteration we have

$$\left(\begin{array}{c} \underset{m1}{\hat{b}_i} \\ \hline \underset{(n-m)1}{\hat{b}_i} \end{array}\right) = \left(\begin{array}{c} \underset{m(M-m)}{\tilde{\Phi}_3} \cdot \underset{(M-m)1}{z_{i-1}} + \underset{mm}{\tilde{\Phi}_1} \cdot \underset{m1}{h} \\ \hline \underset{(n-m)(M-m)}{\tilde{\Phi}_4} \cdot \underset{(M-m)1}{z_{i-1}} + \underset{(n-m)m}{\tilde{\Phi}_2} \cdot \underset{m1}{h} \end{array}\right) \tag{9}$$

If the number of values to which special importance is attached, in the first part of $\underset{m1}{h}$, is greater than the number of coefficients, say m = N where N $\gg$ n then the part of (9) below the partition does not exist and (9) becomes

$$\underset{n1}{\hat{b}i} = \underset{n(M-N)}{\tilde{\Phi}_3} \cdot \underset{(M-N)1}{z_{i-1}} + \underset{nN}{\tilde{\Phi}_1} \cdot \underset{N1}{h} \tag{10}$$

At each stage we have

$$\underset{M1}{z_{i-1}} = \underset{Mn}{\Phi} \cdot \underset{n1}{\hat{b}_{i-1}} \tag{11}$$

i.e.

$$\left(\begin{array}{c} \underset{N1}{z_{i-1}} \\ \hline \underset{(M-N)1}{z_{i-1}} \end{array}\right) = \left(\begin{array}{c} \underset{Nn}{\Phi_1} \\ \hline \underset{(M-N)n}{\Phi} \end{array}\right) \cdot \underset{n1}{\hat{b}_{i-1}} \tag{12}$$

and from (12) we get

$$\underset{N1}{z_{i-1}} = \underset{Nn}{\Phi_1} \cdot \underset{n1}{\hat{b}_{i-1}} \tag{13}$$

and

$$\underset{(M-N)1}{\underline{z}_{i-1}} = \underset{(M-N)n}{\Phi_3} \cdot \underset{n1}{\hat{\underline{b}}_{i-1}} \tag{14}$$

We can now substitute from (14) into (11) to get

$$\underset{n1}{\hat{\underline{b}}_i} = \underset{n(M-N)}{\tilde{\Phi}_3} \cdot \underset{(M-N)n}{\Phi_3} \cdot \underset{n1}{\hat{\underline{b}}_{i-1}} + \underset{nN}{\tilde{\Phi}_1} \cdot \underset{N1}{\underline{h}} \tag{15}$$

If (15) converges, then after sufficient iterations we must end up with

$$\hat{\underline{b}}_i = \tilde{\Phi}_3 \cdot \Phi_3 \cdot \hat{\underline{b}}_i + \tilde{\Phi}_1 \cdot \underline{h} \tag{16}$$

i.e.

$$(\underset{nn}{I} - \tilde{\Phi}_3 \cdot \Phi_3) \cdot \hat{\underline{b}}_i = \tilde{\Phi}_1 \cdot \underline{h} \tag{17}$$

but because

$$\underset{nM}{\tilde{\Phi}} \cdot \underset{Mn}{\Phi} = \underset{nn}{I} \tag{18}$$

the partitioning

$$\underset{Mn}{\Phi} = \begin{pmatrix} \underset{Nn}{\Phi_1} \\ \hline \underset{(M-N)n}{\Phi_3} \end{pmatrix} \tag{19}$$

yields

$$(\tilde{\Phi}_1 \mid \tilde{\Phi}_3) \cdot \begin{pmatrix} \Phi_1 \\ \hline \Phi_3 \end{pmatrix} = \underset{nn}{I}$$

i.e.

$$\tilde{\Phi}_1 \cdot \Phi_1 + \tilde{\Phi}_3 \cdot \Phi_3 = \underset{nn}{I} \tag{20}$$

whence

$$\underset{nn}{I} - \tilde{\Phi}_3 \cdot \Phi_3 = \tilde{\Phi}_1 \cdot \Phi_1 \tag{21}$$

Then, using (21), from (17) we get

$$\underset{nN}{\tilde{\Phi}_1} \cdot \underset{Nn}{\Phi_1} \cdot \underset{n1}{\hat{\underline{b}}_i} = \underset{nN}{\tilde{\Phi}_1} \cdot \underset{N1}{\underline{h}} \tag{22}$$

But (22) is recognizable as the least-squares normal equations yielding the solution of the problem

$$\underset{Nn}{\Phi_1} \cdot \underset{n1}{\underline{b}_i} = \underset{N1}{\underline{h}} + \underset{N1}{\underline{\epsilon}} \tag{23}$$

and so, with $m = N \geq n$ the $\hat{\underline{b}}_i$ obtained after a sufficient number of iterations tends towards that for the least-sqaures fitting of the nominated values in the first part of the data vector $\underline{h}$. The solution is thus independent of the remaining values in the second part of the data vector $\underline{h}$.

We now return to the general expression (9) and deal with the case where the nominated values in the first part of the data vector are fewer than the number of fitting functions, i.e. the case $m < n$. Take the expression (18) and partition it to get

$$\left(\begin{array}{c|c} \underset{mm}{\tilde{\tilde{\Phi}}_1} & \underset{m(M-m)}{\tilde{\tilde{\Phi}}_3} \\ \hline \underset{(n-m)m}{\tilde{\tilde{\Phi}}_2} & \underset{(n-m)(M-m)}{\tilde{\tilde{\Phi}}_4} \end{array}\right) \cdot \left(\begin{array}{c|c} \underset{mm}{\Phi_1} & \underset{m(n-m)}{\Phi_2} \\ \hline \underset{(M-m)m}{\Phi_3} & \underset{(M-m)(n-m)}{\Phi_4} \end{array}\right) = \underset{nn}{I} = \left(\begin{array}{c|c} \underset{mm}{I} & \underset{m(n-m)}{O} \\ \hline \underset{(n-m)m}{O} & \underset{(n-m)(n-m)}{I} \end{array}\right) \tag{24}$$

(24) then yields the following equations

$$\tilde{\tilde{\Phi}}_1 \cdot \Phi_1 + \tilde{\tilde{\Phi}}_3 \cdot \Phi_3 = \underset{mm}{I} \tag{25}$$

$$\tilde{\tilde{\Phi}}_1 \cdot \Phi_2 + \tilde{\tilde{\Phi}}_3 \cdot \Phi_4 = \underset{m(n-m)}{O} \tag{26}$$

$$\tilde{\tilde{\Phi}}_2 \cdot \Phi_1 + \tilde{\tilde{\Phi}}_4 \cdot \Phi_3 = \underset{(n-m)m}{O} \tag{27}$$

$$\tilde{\tilde{\Phi}}_2 \cdot \Phi_2 + \tilde{\tilde{\Phi}}_4 \cdot \Phi_4 = \underset{(n-m)(n-m)}{I} \tag{28}$$

At each stage, we have

$$\underset{M1}{\underline{z}_{i-1}} = \underset{Mn}{\Phi} \cdot \underset{n1}{\hat{\underline{b}}_{i-1}} \tag{29}$$

i.e.

$$\left(\begin{array}{c} \underset{m1}{\underline{z}_{i-1}} \\ \hline \underset{(M-m)1}{\underline{z}_{i-1}} \end{array}\right) = \left(\begin{array}{c|c} \underset{mm}{\Phi_1} & \underset{m(n-m)}{\Phi_2} \\ \hline \underset{(M-m)m}{\Phi_3} & \underset{(M-m)(n-m)}{\Phi_4} \end{array}\right) \cdot \left(\begin{array}{c} \underset{m1}{\hat{\underline{b}}_{i-1}} \\ \hline \underset{(n-m)1}{\hat{\underline{b}}_{i-1}} \end{array}\right) \tag{30}$$

which gives the two equations

$$\underset{m1}{\underline{z}_{i-1}} = \underset{mm}{\Phi_1} \cdot \underset{m1}{\hat{\underline{b}}_{i-1}} + \underset{m(n-m)}{\Phi_2} \cdot \underset{(n-m)1}{\hat{\underline{b}}_{i-1}} \tag{31}$$

$$\underset{(M-m)1}{\underline{z}_{i-1}} = \underset{(M-m)m}{\Phi_3} \cdot \underset{m1}{\hat{\underline{b}}_{i-1}} + \underset{(M-m)(n-m)}{\Phi_4} \cdot \underset{(n-m)1}{\hat{\underline{b}}_{i-1}} \tag{32}$$

and now, using (31) and (32), eqn (9) can be expressed as

$$\left(\begin{array}{c} \underset{m1}{\hat{\underline{b}}_i} \\ \hline \underset{(n-m)1}{\hat{\underline{b}}_i} \end{array}\right) = \left(\begin{array}{c} \tilde{\tilde{\Phi}}_3 \cdot \Phi_3 \cdot \underset{m1}{\hat{\underline{b}}_{i-1}} + \tilde{\tilde{\Phi}}_3 \cdot \Phi_4 \cdot \underset{(n-m)1}{\hat{\underline{b}}_{i-1}} + \tilde{\tilde{\Phi}}_1 \cdot \underline{h} \\ \hline \tilde{\tilde{\Phi}}_4 \cdot \Phi_3 \cdot \underset{m1}{\hat{\underline{b}}_{i-1}} + \tilde{\tilde{\Phi}}_4 \cdot \Phi_4 \cdot \underset{(n-m)1}{\hat{\underline{b}}_{i-1}} + \tilde{\tilde{\Phi}}_2 \cdot \underline{h} \end{array}\right) \tag{33}$$

and if this converges it must, after a sufficient number of iterations, come to

$$\left(\begin{array}{c} \underset{m1}{\hat{\underline{b}}_i} \\ \hline \underset{(n-m)1}{\hat{\underline{b}}_i} \end{array}\right) = \left(\begin{array}{c} \tilde{\tilde{\Phi}}_3 \cdot \Phi_3 \cdot \underset{m1}{\hat{\underline{b}}_i} + \tilde{\tilde{\Phi}}_3 \cdot \Phi_4 \cdot \underset{(n-m)1}{\hat{\underline{b}}_i} + \tilde{\tilde{\Phi}}_1 \cdot \underline{h} \\ \hline \tilde{\tilde{\Phi}}_4 \cdot \Phi_3 \cdot \underset{m1}{\hat{\underline{b}}_i} + \tilde{\tilde{\Phi}}_4 \cdot \Phi_4 \cdot \underset{(n-m)1}{\hat{\underline{b}}_i} + \tilde{\tilde{\Phi}}_2 \cdot \underline{h} \end{array}\right) \tag{34}$$

To see what (34) implies for $m < n$, consider the problem of fitting the nominated values in the first part of the $\underline{h}$ vector exactly and the $\underline{z}_o$ values in the second part of the $\underline{h}$ vector by least-squares by one and the same coefficient vector. We have to solve

$$\left(\begin{array}{c|c} \underset{m\,m}{\Phi_1} & \underset{m(n-m)}{\Phi_2} \\ \hline \underset{(M-m)\,m}{\Phi_3} & \underset{(M-m)(n-m)}{\Phi_4} \end{array}\right) \cdot \left(\begin{array}{c} \underline{b}_i \\ \hline \underline{b}_i \end{array}\right) = \left(\begin{array}{c} \underline{h} \\ \hline \underline{z}_o + \underline{\tau} \end{array}\right) \qquad (35)$$

i.e. to solve simultaneously the two equations

$$\Phi_1 \cdot \underline{b}_i + \Phi_2 \cdot \underline{b}_i = \underline{h} \qquad \text{exactly} \qquad (36)$$

and

$$\Phi_3 \cdot \underline{b}_i + \Phi_4 \cdot \underline{b}_i = \underline{z}_o + \underline{\tau} \qquad \text{least-squares} \qquad (37)$$

It is relatively easy to show that if (36) is subsituted into (34) then (34) is satisfied. To do this take $\tilde{\Phi}_1 \cdot$ and $\tilde{\Phi}_2 \cdot$ through (36) and substitute for $\tilde{\Phi}_1 \cdot \underline{h}$ and $\tilde{\Phi}_2 \cdot \underline{h}$ in (34). Then taking (25) to (28) into account (34) is found to be satisfied.

To get the least-squares solution to (37) rewrite it as

$$(\Phi_3 \mid \Phi_4) \cdot \left(\begin{array}{c} \underline{b}_i \\ \hline \underline{b}_i \end{array}\right) = \underline{z}_o + \underline{\tau} \qquad (38)$$

The normal equations are then

$$\left(\begin{array}{c} \tilde{\Phi}_3 \\ \hline \tilde{\Phi}_4 \end{array}\right) \cdot (\Phi_3 \mid \Phi_4) \cdot \left(\begin{array}{c} \hat{b}_i \\ \hline \hat{b}_i \end{array}\right) = \left(\begin{array}{c} \tilde{\Phi}_3 \\ \hline \tilde{\Phi}_4 \end{array}\right) \cdot \underline{z}_o \qquad (39)$$

and may be rewritten as

$$\left(\begin{array}{c|c} \tilde{\Phi}_3 \cdot \Phi_3 & \tilde{\Phi}_3 \cdot \Phi_4 \\ \hline \tilde{\Phi}_4 \cdot \Phi_3 & \tilde{\Phi}_4 \cdot \Phi_4 \end{array}\right) \cdot \left(\begin{array}{c} \hat{b}_i \\ \hline \hat{b}_i \end{array}\right) = \left(\begin{array}{c} \tilde{\Phi}_3 \\ \hline \tilde{\Phi}_4 \end{array}\right) \cdot \underline{z}_o \qquad (40)$$

i.e.

$$\left(\begin{array}{c} (\tilde{\Phi}_3 \cdot \Phi_3) \cdot \underline{b}_i + (\tilde{\Phi}_3 \cdot \Phi_4) \cdot \underline{b}_i \\ \hline (\tilde{\Phi}_4 \cdot \Phi_3) \cdot \underline{b}_i + (\tilde{\Phi}_4 \cdot \Phi_4) \cdot \underline{b}_i \end{array}\right) = \left(\begin{array}{c} \tilde{\Phi}_3 \cdot \underline{z}_o \\ \hline \Phi_4 \cdot \underline{z}_o \end{array}\right) \qquad (41)$$

and if we substitute (41) into (34) there results

$$\left(\begin{array}{c} \underline{b}_i \\ \hline \underline{b}_i \end{array}\right) = \left(\begin{array}{c} \tilde{\Phi}_3 \cdot \underline{z}_o + \tilde{\Phi}_1 \cdot \underline{h} \\ \hline \tilde{\Phi}_4 \cdot \underline{z}_o + \tilde{\Phi}_2 \cdot \underline{h} \end{array}\right) \qquad (42)$$

Finally, it is now necessary to check that, if we substitute from (36) for $\tilde{\Phi}_1 \cdot \underline{h}$ and $\tilde{\Phi}_2 \cdot \underline{h}$, (42) is still satisfied. After substitution we have

$$\left(\frac{\underline{b}_{i\,m1} - \tilde{\Phi}_1 \cdot (\Phi_1 \cdot \underline{b}_{i\,m1} + \Phi_2 \cdot \underline{b}_{i\,(n-m)1})}{\underline{b}_{i\,(n-m)1} - \tilde{\Phi}_2 \cdot (\Phi_1 \cdot \underline{b}_{i\,m1} + \Phi_2 \cdot \underline{b}_{i\,(n-m)1})} \right) = \left(\frac{\tilde{\Phi}_3 \cdot \underline{z}_{o\,(M-m)1}}{\tilde{\Phi}_4 \cdot \underline{z}_{o\,(M-m)1}} \right) \qquad (43)$$

By using (25) and (28) this can be put into the form

$$\left(\frac{(\tilde{\Phi}_3 \cdot \Phi_3) \cdot \underline{b}_{i\,m1} - (\tilde{\Phi}_1 \cdot \Phi_2) \cdot \underline{b}_{i\,(n-m)1}}{(\tilde{\Phi}_4 \cdot \Phi_4) \cdot \underline{b}_{i\,(n-m)1} - (\tilde{\Phi}_2 \cdot \tilde{\Phi}_1) \cdot \underline{b}_{i\,m1}} \right) = \left(\frac{\tilde{\Phi}_3 \cdot \underline{z}_{o\,(M-m)1}}{\tilde{\Phi}_4 \cdot \underline{z}_{o\,(M-m)1}} \right) \qquad (44)$$

and then by using (26) and (27), (44) becomes

$$\left(\frac{(\tilde{\Phi}_3 \cdot \Phi_3) \cdot \underline{b}_{i\,m1} + (\tilde{\Phi}_3 \cdot \Phi_4) \cdot \underline{b}_{i\,(n-m)1}}{(\tilde{\Phi}_4 \cdot \Phi_3) \cdot \underline{b}_{i\,m1} + (\tilde{\Phi}_4 \cdot \Phi_4) \cdot \underline{b}_{i\,(n-m)1}} \right) = \left(\frac{\tilde{\Phi}_3 \cdot \underline{z}_{o\,(M-m)1}}{\tilde{\Phi}_3 \cdot \underline{z}_{o\,(M-m)1}} \right) \qquad (45)$$

But this is the same as (41), Q.E.D.

To sum up, the situation is -

$m \geq n$	. more nominated values than coefficients

The nominated values are fitted least-squares and the fitting is independent of the remaining data values

$m < n$	more coefficients than nominated values

The nominated values are fitted exactly and the remaining data values are fitted least-squares

This is the situation when the process has gone through a sufficient number of iterations.

3. Comments

It will be apparent to the reader that the $\underline{z}_o$ field need not be a least-squares fitting. Any preliminary fitting will serve as a starting point. It is not intended to suggest that the above procedure is the only means by which the above effects may be adduced. The m n mode has obvious application in an interactive situation where, following a study of a preliminary fitting, an experimenter may wish to insert bogus values in place of some of the original data values. Similarly the m > n affords a way of studying the effect of leaving some data values out altogether.

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